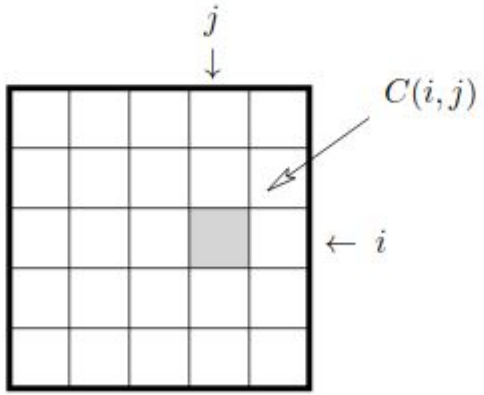
Dinitz! Graphs! Coloring!

May 2020

The Dinitz Problem

raised by Jeff Dinitz in 1978, which defied all attacks until its astonishingly simple solution by Fred Galvin fifteen years later.

> *Consider n^2 cells arranged in an $(n \times n)$-square, and let (i, j) denote the cell in row i and column j. Suppose that for every cell (i, j) we are given a set $C(i, j)$ of n colors.*
> *Is it then always possible to color the whole array by picking for each cell (i, j) a color from its set $C(i, j)$ such that the colors in each row and each column are distinct?*



Kind of like Sudoku where numbers are colors

Trival Cases for *n* x *n* grid

All color sets are different (e.g n^2 nodes, n^2 colors)

$$\forall x, x' : x \neq x', C(x) \cap C(x') = \emptyset$$

All color sets are the same and there are n colors (Latin Square)

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Works with any arbitrary initial permutation

Can't do it with < n colors

More colors is more difficult as one cannot start arbitrarily

$\{1,2\}$	$\{2,3\}$
$\{1,3\}$	$\{2,3\}$

1. Start with the first row being 1, 2
2. Forced to choose 3 for cell 1, 0
3. Choosing 2 or 3 for cell 1, 1 does not work

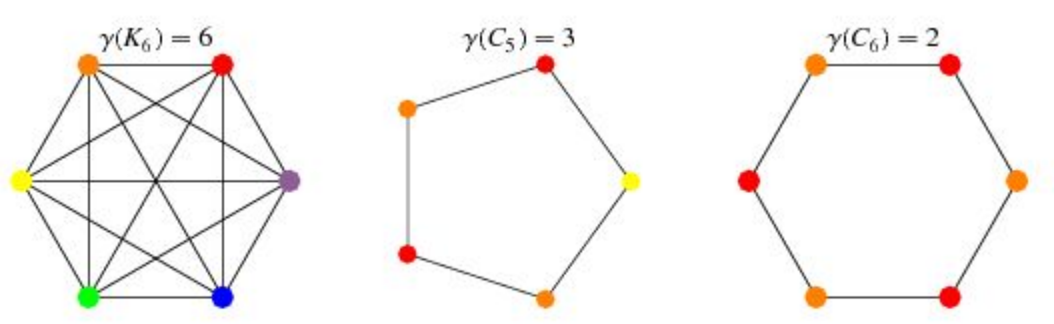
Smaller set of solutions if they even exist (I think)

Undirected Graph Definitions: G(V, E)

G(*V*, *E*) is a graph with a set of vertices/nodes (*V*) and edges (*E*)

X(*G*) is the *chromatic number* of *G*:

Smallest number of independent sets (set with vertices that do not share edges) that partition *V* (cover all of *V*)



E.g Smallest amount of colors needed to color the graph (ignore color sets for each vertex

Undirected Graph Definitions: G(V, E)

List coloring is a mapping c: V → $U_{v \in V}$C(v) such that c(v) \in C(v) and for all v, v' in E, c(v) != c(v')

E.g Given each node has a color set, an assignment that picks from each node's color set such that adjacent nodes don't share the same color

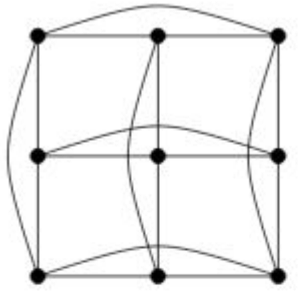
X_l(G) is the list chromatic number: smallest k s.t for all color sets of size k over V (C(v_1), ..., C(v_N)), there exists a list coloring

Note that *X_l(G) >= X(G)* since same-color set coloring is just a specific choice of color set. *X_l(G)* is the smallest k for all color sets (of size k), including the super messed up ones

Dinitz in Graph Language

S_n is the grid with n rows/cols, squares as nodes and edges shared by nodes in the same row or column

$X_l(S_n) = n$? (Why can't this be less than n? $X(S_n) = n$)



The graph S_3

raised by Jeff Dinitz in 1978, which defied all attacks until its astonishingly simple solution by Fred Galvin fifteen years later.

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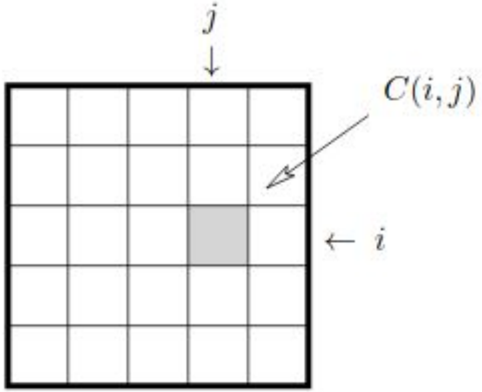
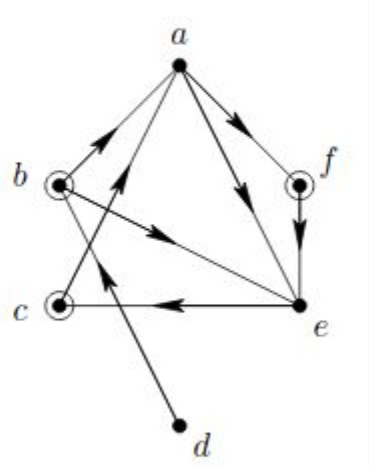


Subgraphs, Directed Graphs notation

Subgraph: If *A* is a subset of *V*, G_A is the subgraph formed from *A* and all the edges from *G* containing *A*

H is an induced subgraph of *G* if there exists *A*, a subset of *V* and $H = G_A$

G(*V, E*) *(i.e BOLD G)* is a directed graph, edges have direction.
$d^+(v)$ is the outdegree, $d^-(v)$ is the indegree and $d^+(v) + d^-(v) = d(v)$



K, a subset of V, is a kernel if:
i) K is independent in G (not ***G****)*
ii) for all u \notin K, there exists a v in K s.t u → v

Lemma 1 Proof

Lemma 1. *Let $\vec{G} = (V, E)$ be a directed graph, and suppose that for each vertex $v \in V$ we have a color set $C(v)$ that is larger than the outdegree, $|C(v)| \geq d^+(v) + 1$. If every induced subgraph of $\vec{G}$ possesses a kernel, then there exists a list coloring of G with a color from $C(v)$ for each v.*

Why will this be useful? (Dinitz to Lemma 1 took 14 yrs, Lemma 1 to end took 1 yr)

- we have S_n which is undirected so to use this we'll need to convert S_n into a directed graph (find a direction for the edges, an orientation)
- with "enough" colors, there exists a list coloring

after Lemma 1, we still need to

- find an orientation such that $d^+(v) <= n - 1$ (create the directed version $\mathbf{S_n}$)
- show every induced subgraph of $\mathbf{S_n}$ has a kernel

Lemma 1 Proof

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Backwards induction?

If $|V| = 1$, nothing to prove, so assume $|V| > 1$.

Start of loop

Choose an arbitrary color from the union of all color sets, *c*. Form the set of nodes with c in their respective color sets, A(c).

Induce the subgraph on A(c) from G → $G_{A(c)}$. By hypothesis, there exists a kernel K(c) on that subgraph. Color all the nodes in K(c) by c (they are independent so not adjacent).

Create a new graph/induce a subgraph, G', from V\K(c) with new color sets C'(v) = C(v)\{c}.

End loop

The condition $|C'(v)| >= d^+(v) + 1$ still holds for v in A(c)\K(c) since we deleted at least one of their old kernel neighbors and a single color (RHS decreases more than LHS). For v \notin A(c), their color sets stay the same and their outdegree weakly decreases so the condition still holds for them as well. Note that |G'| < |G|, so we're done.

Where are we?

Lemma 1. *Let $\vec{G} = (V, E)$ be a directed graph, and suppose that for each vertex $v \in V$ we have a color set $C(v)$ that is larger than the outdegree, $|C(v)| \geq d^+(v) + 1$. If every induced subgraph of $\vec{G}$ possesses a kernel, then there exists a list coloring of G with a color from $C(v)$ for each v.*

We are done if:

- We find an orientation for S_n (convert S_n into the directed version $\mathbf{S_n}$) that also satisfies the outdegree condition (n >= $d^+(v) + 1$) AND
- we prove that every induced subgraph of $\mathbf{S_n}$ possesses a kernel

Note: last line of Lemma 1 is there exists a list coloring of the undirected graph.

Detour into Bipartite Graphs

G = (X ∪ Y, E), where edges connect x and y, not x and x' nor y and y'

Intuition: X is the set of Men and Y is the set of Women and an edge is a pairing of a man and woman

A *matching* is a bipartite graph where none of the edges share an end vertex. E.g a set of marriages where no bigamy (man has multiple wives or woman has multiple husbands).

Adding preferences...suppose each node, v (in X or Y) has an ordering their adjacent nodes, N(v) = $\{z_1 > z_2 > \ldots > z_{d(v)}\}$

A *stable matching* is a matching where no 2 nodes are each better off by forming another marriage.
I.e for all uv in E\M (a feasible match between u in X and v in Y), either uy in M and y > v in N(u) (u is matched with someone they prefer) or xv in M and x > u in N(v) (v is matched with someone they prefer)

Stable Matching Always Exists

Proof:

Set R = X and all strings for females are empty

Loop until R is empty

1. All men, u in R, propose to their *current* top choice (no going backwards)
2. If a girl receives more than one proposal, she chooses the top pick among the current proposals and puts him on a "string"
3. Men rejected with no options left die, all the other rejected men go into the reservoir R (update R)
4. Repeat

This terminates as each loop has some men go strictly forward through their list of choices. The is stable because

Suppose uv in E but uv not in M. Either:

Case 1: u never proposed to v
- he stopped before getting to v in favor of someone else (exists y in Y s.t uy in M with y > v in N(u))
Case 2: u did propose to v
- v rejected the proposal in favor of someone else (exists x in X s.t xv in M with x > v in N(v))

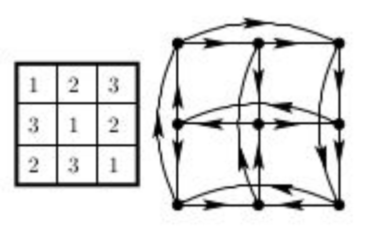
Dinitz Proof: Lemma 1 + Lemma 2

Step 1: Find an orientation such that $d^+(i, j) <= n - 1$

Step 2: For this new directed graph, show that every induced subgraph has a kernel

Setup:

Vertices are denoted i, j (row i, col j). Therefore (i, j) and (r, s) are adjacent if i = r or j = s.



Step 1:

Setup a Latin square.
For each of the rows, a node u has an edge to v in the row if L(u) < L(v) (this sets up horizontal edges)
For each of the columns, a node u has an edge to v in the col if L(u) > L(v) (this sets up vertical edges)

Note that a given node has n - 1 other nodes in the same row and n - 1 other nodes in the same col, so 2n - 2. Because the rows and columns point in <, > fashion, each node has outdegree of n - 1 (half of 2n - 2).

Therefore $d^+(i, j) <= n - 1$ holds!

Dinitz Proof: Lemma 1 + Lemma 2

Step 2: For this new directed graph, show that every induced subgraph has a kernel

Take some subset of the nodes A (e.g {(1, 1), (1, 2), {4, 3)}

Let X be the set of rows and Y be the set of columns e.g X = {1, ..., n} and Y = {1, ..., n}

Create the bipartite graph G = (X ∪ Y, A) (A denotes connection between rows and cols)

Use the orientation (directedness of edges) to create preferences ala Marriage (men are rows and women are columns).
j > j' in N(i) if (i, j) → (i, j') in $\mathbf{S_n}$. Similarly i > i' in N(j) if (i, j) → (i', j) in $\mathbf{S_n}$.

Lemma 2 says there exists a stable matching, M. M, a subset of A, is the kernel! Why?

1) M is independent since it is a matching (i.e no common endpoints in M/row j only appears once if at all)
2) Take an edge outside the kernel, (i, j) in A\M, by stable matching either
    a) There exists j' s.t (i, j') in M and j' > j which means (i, j) → (i, j') in M (because preferences were constructed from the directedness of $\mathbf{S_n}$.
    b) There exists i' s.t (i', j) in M and i' > i which means (i, j) → (i', j) in M